

LOCAL EXISTENCE OF A FOURTH ORDER DISPERSIVE CURVE FLOW ON LOCALLY HERMITIAN SYMMETRIC SPACES AND THE APPLICATION

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ABSTRACT. This paper is concerned with a fourth order nonlinear dispersive partial differential equation for closed curve flow on a Kähler manifold. The main results is that the initial value problem has a solution locally in time if the Kähler manifold is a compact locally hermitian symmetric space. The proof is based on the geometric energy method combined with a nice gauge transformation to eliminate the loss of derivatives. Interestingly, the results can be applied to construct a generalized bi-Schrödinger flow proposed by Ding and Wang. The assumption on the manifold plays a crucial role both to enjoy a good solvable structure of the problem and to reduce the generalized bi-Schrödinger flow equation to the one considered in the present paper.

1. INTRODUCTION

Let N be a compact Kähler manifold with the complex structure J and with a Kähler metric h , and let $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ being the one-dimensional flat torus. We consider the initial value problem of the form

$$u_t = a J_u \nabla_x^3 u_x + \lambda J_u \nabla_x u_x + b R(\nabla_x u_x, u_x) J_u u_x + c R(J_u u_x, u_x) \nabla_x u_x \quad \text{in } \mathbb{R} \times \mathbb{T}, \quad (1.1)$$

$$u(0, x) = u_0(x) \quad \text{in } \mathbb{T}, \quad (1.2)$$

where the solution is a map $u = u(t, x) : \mathbb{R} \times \mathbb{T} \rightarrow N$ being a closed curve flow on N and the initial function is given by $u_0 = u_0(x) : \mathbb{T} \rightarrow N$. For the map u , the velocity vector field in t and x is respectively denoted by $u_t = du(\frac{\partial}{\partial t})$ and $u_x = du(\frac{\partial}{\partial x})$ where du is the differential of u , the covariant derivative along u in x is by ∇_x , the complex structure at $u \in N$ is by J_u , and the Riemannian curvature tensor on (N, h) is by $R = R(\cdot, \cdot)$. Moreover, a, b, c, λ are real constants and $a \neq 0$ is supposed so that (1.1) is handled as a fourth order nonlinear dispersive partial differential equation.

The equation (1.1) is derived by generalizing a two-sphere-valued physical model. Lakshmanan, Porsezian, and Daniel [15] studied the continuum limit of the Heisenberg spin chain systems with biquadratic exchange interactions and formulated the model equation, which reads

$$u_t = u \wedge [a_1 \partial_x^3 u_x + \{1 + a_2 (u_x, u_x)\} \partial_x u_x + 2a_2 (\partial_x u_x, u_x) u_x], \quad (1.3)$$

where $u : \mathbb{R} \times \mathbb{T} \rightarrow \mathbb{S}^2 \subset \mathbb{R}^3$ is the solution, \mathbb{S}^2 denotes the two-dimensional unit sphere centered at the origin, ∂_x is the partial differential operator in x acting on \mathbb{R}^3 -valued functions, (\cdot, \cdot) and \wedge denotes the inner and the exterior product in \mathbb{R}^3 respectively, and $a_1 \neq 0, a_2 \in \mathbb{R}$ are real physical constants. Moreover, (1.3) is known to arise in connection with the equation derived by Fukumoto and Moffatt [9, 10] to model the vortex filament in an incompressible perfect fluid in \mathbb{R}^3 . As is found in Section 2.2, (1.3) is reformulated as (1.1) where $N = \mathbb{S}^2$ with the complex structure $J_u = u \wedge$ being the $\pi/2$ -degree rotation on $T_u \mathbb{S}^2$ and with $h(\cdot, \cdot) = (\cdot, \cdot)$ being the canonical metric induced from the Euclidean metric in \mathbb{R}^3 and $a = a_1, b = 5a_1 - 2a_2, c = -6a_1 + 3a_2$, and $\lambda = 1$.

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It is to be commented that another geometric generalization of (1.3) has been proposed in [19]. The equation can be formulated by

$$u_t = b_1 J_u \nabla_x^3 u_x + b_2 J_u \nabla_x u_x + b_3 h(u_x, u_x) J_u \nabla_x u_x + b_4 h(\nabla_x u_x, u_x) J_u u_x \quad (1.4)$$

for $u = u(t, x) : \mathbb{R} \times \mathbb{T} \rightarrow N$, where (N, J, h) is a compact Kähler manifold, and $b_1 \neq 0$, $b_2, b_3, b_4 \in \mathbb{R}$ are real constants. As in [19], the equation (1.3) is actually generalized also as (1.4) where $N = \mathbb{S}^2$, $b_1 = a_1$, $b_2 = 1$, $b_3 = a_2 - a_1$, and $b_4 = -5a_1 + 2a_2$. In fact, (1.1) and (1.4) are essentially the same, if (N, J, h) is a Riemann surface with constant Gaussian curvature. See Section 2.2 for the detail.

We are interested in the solvability of the initial value problem for (1.1), (1.3), and (1.4) in the framework of a Sobolev space. The main difficulty comes from the so-called loss of derivatives, which prevents the classical energy method from working to construct a time-local solution. More concretely, by the seemingly bad structure of first and second order terms in the equation, the estimate for the standard Sobolev norm of the solution can not be closed only by the integration by parts. In general, to clarify the solvability of the initial value problem for a dispersive partial differential equation, the structure of the lower order terms in the equation is to be analyzed in detail. The structure obviously becomes more complicated as the spatial order of the equation becomes higher (See, e.g., [3, 17] for details.). In other words, the solvable structure of (1.1), (1.3), and (1.4) seems to be connected with the geometric setting of (N, J, h) essentially.

We state here the known results in the direction. Guo, Zeng, and Su [11] investigated the \mathbb{S}^2 -valued model (1.3) and showed time-local existence of a weak solution to the initial value problem imposing that $2a_2 = 5a_1$. In the proof, two types of conservation laws (which do not hold if $2a_2 \neq 5a_1$) works effectively to overcome the difficulty of loss of derivatives. In recent years, for both (1.3) and (1.4), the loss of derivatives have been found to be eliminated completely by the energy method combined with a kind of nice gauge transformation acting on the highest order derivatives of the solution. Indeed, the author [21] studied the initial value problem for (1.3) and showed time-local existence of a smooth solution and the uniqueness without any assumption on $a_1 \neq 0, a_2 \in \mathbb{R}$. Moreover, he showed the time-global existence imposing that $2a_2 = 5a_1$. After that, the author [22] studied the initial value problem for (1.4) and showed time-local existence of a smooth solution and the uniqueness under the assumption that (N, J, h) is a compact Riemann surface with constant Gaussian curvature without any assumption on $b_1 \neq 0, \dots, b_4 \in \mathbb{R}$. To our interest, the necessity in some sense of the assumption on the curvature of (N, h) is pointed out by Chihara [3] from the point of view of the theory of linear dispersive partial differential equations. Incidentally, Chihara and the author [5] studied the initial value problem for (1.4) replacing \mathbb{T} with the real line \mathbb{R} as the spatial domain, and showed time-local existence of a smooth solution and the uniqueness without any assumption on the compact Kähler manifold (N, J, h) and constants $b_1 \neq 0, \dots, b_4 \in \mathbb{R}$. The proof essentially applies the local smoothing effect for the fundamental solution to dispersive partial differential equations on \mathbb{R} to dominate the loss of derivatives. However, the method of the proof is not valid in the case of \mathbb{T} , since the above dispersive smoothing effect is absent on compact spatial domains.

The purpose of the present paper was to clarify the solvability of the initial value problem (1.1)-(1.2) without the dispersive smoothing effect, which is a continuation of the work [21, 22] stated above. The difficulty of loss of derivatives occurs also for (1.1). However, in view of the similarity between (1.1) and (1.4), it seems natural to expect positive results under a suitable geometric assumption on (N, J, h) . Indeed, we have found a sufficient condition on (N, J, h) to ensure the existence of a time-local smooth solution. More precisely, the main results is stated as follows:

Theorem 1.1. *Suppose that (N, J, h) is a compact Kähler manifold satisfying $\nabla R = 0$. Let k be an integer satisfying $k \geq 4$. Then for any $u_0 \in C(\mathbb{T}; N)$ satisfying $u_{0x} \in H^k(\mathbb{T}; TN)$, there exists $T = T(\|u_{0x}\|_{H^4(\mathbb{T}; TN)}) > 0$ such that (1.1)-(1.2) has a solution $u \in C([-T, T] \times \mathbb{T}; N)$ satisfying $u_x \in L^\infty(-T, T; H^k(\mathbb{T}; TN)) \cap C([-T, T]; H^{k-1}(\mathbb{T}; TN))$.*

Notation. Let $\phi : \mathbb{T} \rightarrow N$. The set of all vector fields along ϕ is denoted by $\Gamma(\phi^{-1}TN)$. The class $H^m(\mathbb{T}; TN)$ for $m = 0, 1, 2, \dots$ consists of all elements $V \in \Gamma(\phi^{-1}TN)$ satisfying

$$\|V\|_{H^m(\mathbb{T}; TN)} := \sum_{\ell=0}^m \int_{\mathbb{T}} h(\nabla_x^\ell V(x), \nabla_x^\ell V(x)) dx < \infty.$$

In particular, $H^0(\mathbb{T}; TN) = L^2(\mathbb{T}; TN)$ for simplicity.

We state a contribution of Theorem 1.1. Recall that (1.1) is nothing but (1.4) as far as (N, J, h) is a compact Riemann surface with constant Gaussian curvature and the case has been already investigated in [21] and [22]. Hence what seems to be interesting or meaningful is the other cases. Fortunately, Theorem 1.1 actually includes such cases. Indeed, in Riemannian geometry, a Kähler manifold (N, J, h) that satisfies $\nabla R = 0$ is called a locally hermitian symmetric space, and the class of compact locally hermitian symmetric spaces is known to be purely wider than that of compact Riemann surfaces with constant Gaussian curvature. For example, Theorem 1.1 includes the case where (N, J, h) is a compact Kähler manifold with constant holomorphic sectional curvature. Such a manifold is not necessarily a Riemann surface with constant sectional curvature. Incidentally, the uniqueness of the solution has been proved in [22] for (1.4) imposing the regularity $k \geq 6$. The proof uses an isometric embedding of the manifold into the higher dimensional Euclidean space to evaluate the difference of two solutions. Although the author expects that the uniqueness holds also for (1.1)-(1.2), the procedure requires a lengthy computation and hence we do not pursue in the direction in the present paper.

To prove Theorem 1.1, we apply the geometric energy method combined with a gauge transformation. The geometric classical energy method without gauge transformations is known to work in the study of the Schrödinger map flow equation, a second order dispersive equation, for maps into Kähler manifolds. Indeed, under the setting, many local and global existence results have been established. See, e.g., the pioneering work by Koiso [14], Chang, Shatah, and Uhlenbeck [1], Ding and Wang [7], Nahmod et al. [18], McGahagan [16], Rodnianski, Rubinstein, and Staffilani [23], Kenig et al. [13], and references therein. On the other hand, the geometric energy method with a gauge transformation was first introduced by Chihara [2]. He showed time-local existence and the uniqueness of a solution to the initial value problem for the Schrödinger map flow equation without the Kähler condition on the manifold. The method has been applied to study a fourth order dispersive flow equation (1.4) in [5, 22] as stated above as well as a third order dispersive flow equation in [4, 20]. We slightly modify the one used in [22] to prove Theorem 1.1.

More precisely, the idea of deciding the form of the gauge transformation used in the present paper is as follows. Suppose u is a smooth solution to (1.1)-(1.2). If $k \geq 4$, the partial differential equation satisfied by $\nabla_x^k u_x$ is found to be described by

$$(\nabla_t - a J_u \nabla_x^4 - d_1 P_1 \nabla_x^2 - d_3 P_2 \nabla_x) \nabla_x^k u_x = \mathcal{O} \left(\sum_{m=0}^{k+2} |\nabla_x^m u_x|_h \right), \quad (1.5)$$

where $|\cdot|_h = \{h(\cdot, \cdot)\}^{1/2}$, d_1 and d_3 are real constants depending on a, b, c, k , and P_1 and P_2 are defined by

$$P_1 Y = R(Y, J_u u_x) u_x \quad \text{and} \quad P_2 Y = R(J_u \nabla_x u_x, u_x) Y$$

respectively for any $Y \in \Gamma(u^{-1}TN)$. See (3.30) for the detail where d_1 and d_3 are also given exactly. From (1.5), the classical energy estimate for $\|\nabla_x^k u_x\|_{L^2(\mathbb{T}; TN)}^2$ is found to break down because the two operators $d_1 P_1 \nabla_x^2$ and $d_3 P_2 \nabla_x$ cause loss of derivatives. Though the right hand side of (1.5) includes $\nabla_x^2(\nabla_x^k u_x)$ and $\nabla_x(\nabla_x^k u_x)$, no loss of derivatives occur thanks to the assumption $\nabla R = 0$ and the Kähler condition $\nabla J = 0$ on (N, J, h) . Hence it suffices to eliminate the loss of derivatives which come from $d_1 P_1 \nabla_x^2$ and $d_3 P_2 \nabla_x$. For this purpose, inspired by [3, 22], we introduce the gauge transformed function V_k defined by

$$V_k = \nabla_x^k u_x - \frac{e_1}{2a} R(\nabla_x^{k-2} u_x, u_x) u_x + \frac{e_2}{8a} R(J_u u_x, u_x) J_u \nabla_x^{k-2} u_x, \quad (1.6)$$

where e_1 and e_2 are constants to be decided later. Notice that V_k is formally expressed by

$$V_k = \left(I_d - \frac{e_1}{2a} \Phi_1 \nabla_x^{-2} + \frac{e_2}{8a} \Phi_2 \nabla_x^{-2} \right) \nabla_x^k u_x,$$

where I_d is the identity on $\Gamma(u^{-1}TN)$ and Φ_1 and Φ_2 is defined by

$$\Phi_1 Y = R(Y, u_x) u_x \quad \text{and} \quad \Phi_2 = R(J_u u_x, u_x) J_u Y$$

respectively for any $Y \in \Gamma(u^{-1}TN)$. Noting that J_u commutes with Φ_2 and not with Φ_1 , we obtain

$$\left[a J_u \nabla_x^4, -\frac{e_1}{2a} \Phi_1 \nabla_x^{-2} \right] \nabla_x^k u_x = (-e_1 P_1 \nabla_x^2 - e_1 P_2 \nabla_x) \nabla_x^k u_x + \text{harmless terms}, \quad (1.7)$$

$$\left[a J_u \nabla_x^4, \frac{e_2}{8a} \Phi_2 \nabla_x^{-2} \right] \nabla_x^k u_x = -e_2 P_2 \nabla_x \nabla_x^k u_x + \text{harmless terms}, \quad (1.8)$$

where $[\cdot, \cdot]$ denotes the commutator bracket of two operators. Therefore, if we set $e_1 = -d_1$ and $e_2 = d_1 - d_3$, the above two commutators eliminate $d_1 P_1 \nabla_x^2 + d_3 P_2 \nabla_x$ in the partial differential equation satisfied by V_k , and hence the energy estimate for $\|u_x\|_{H^{k-1}(\mathbb{T}; TN)}^2 + \|V_k\|_{L^2(\mathbb{T}; TN)}^2$ works. More precisely, the standard compactness argument with the above energy estimate for the family of parabolic regularized solutions shows the existence of a solution locally in time, which completes the proof.

Let us now turn our attention to one more interesting conclusion in connection with the so-called generalized bi-Schrödinger flow. Recently, Ding and Wang [6] proposed a fourth order dispersive partial differential equation for maps into Kähler manifolds or para-Kähler manifolds, whose solution is called a generalized bi-Schrödinger flow. We recall briefly the definition and their work in [6] restricting to the part which is related to the present paper. Let (M, g) be an m -dimensional Riemannian manifold with a Riemannian metric g and let (N, J, h) be a $2n$ -dimensional Kähler manifold with the complex structure J and a Kähler metric h . Fix $\{e_1, \dots, e_m\}$ as a local frame of (M, g) . In the local frame, g is expressed by $g = (g_{ij})$ and its inverse is by (g^{ij}) . Let $\alpha, \beta, \gamma \in \mathbb{R}$ be constants where $\beta \neq 0$. Then the energy functional $E_{\alpha, \beta, \gamma}(u)$ for smooth maps $u : (M, g) \rightarrow (N, J, h)$ is defined by

$$E_{\alpha, \beta, \gamma}(u) := \alpha E(u) + \beta E_2(u) + \gamma E_*(u).$$

Here, $E(u) = \frac{1}{2} \int_M |du|^2 dv_g$ is the energy functional whose critical points are called harmonic maps and $E_2(u) = \frac{1}{2} \int_M |\tau(u)|^2 dv_g$ is the bi-energy functional whose critical points are called bi-harmonic maps, where $\tau(u)$ is the tension field along u and dv_g is the volume form of (M, g) . One can consult with [8, 12] for their definition. The energy functional $E_*(u)$ is defined by

$$E_*(u) = \int_M \sum_{i,j,k,\ell=1}^m g^{ij} g^{kl} h(R(\nabla_{e_i} u, J_u \nabla_{e_j} u) J_u \nabla_{e_k} u, \nabla_{e_\ell} u) dv_g,$$

where ∇_{e_k} is the covariant derivative on the pull-back bundle $u^{-1}TN$ induced from the Levi-Civita connection on (N, h) , $R(\cdot, \cdot)$ is the Riemannian curvature tensor on (N, h) . The definition

is of significance, in that the function in the above integral is independent of the choice of a local frame $\{e_1, e_2, \dots, e_m\}$. A time-dependent map $u = u(t, x) : (-T, T) \times M \rightarrow N$ is called a generalized bi-Schrödinger flow from (M, g) to (N, J, h) if u satisfies the following Hamiltonian gradient flow equation

$$u_t = J_u \nabla E_{\alpha, \beta, \gamma}(u) \quad (1.9)$$

in $(-T, T) \times M$ for some $T > 0$. For (1.9), the authors in [6] investigated maps from \mathbb{R} to the symmetric space N where N is one of $G_{n,k}$ or G_n^k . Here $G_{n,k}$ is the Kähler Grassmannian manifold of compact type and G_n^k is the Kähler Grassmannian manifold of noncompact type. They showed that (1.9) corresponds to the following more specific form

$$\varphi_t = [\varphi, -\alpha \varphi_{xx} + \beta \varphi_{xxxx} + (4\gamma - 2\beta)(\varphi_x \varphi^{-1} \varphi_x \varphi^{-1} \varphi_x)_x] \quad (1.10)$$

for $\varphi : \mathbb{R} \times \mathbb{R} \rightarrow N$. Furthermore, they reduced (1.10) to a fourth order nonlinear Schrödinger-like matrix equation. See [6] for more details. What seems to be interesting in connection with the present paper is that the equation (1.9) for maps from \mathbb{R} or \mathbb{T} to locally hermitian symmetric spaces is found to be formulated as the equation (1.1). In fact, in Section 2, we formulate (1.9) as (2.6) imposing that (M, g) is an m -dimensional flat torus (\mathbb{T}^m, g_0) and (N, J, h) is a locally hermitian symmetric space. Then the derived equation (2.6) is easily found to be just (1.1) if $m = 1$ (see (2.7)). To obtain the formulation, we apply the method in [13]. Though we demonstrate only the case where $(M, g) = (\mathbb{T}^m, g_0)$, we can obviously obtain the the same equation as (2.6) also in the case (M, g) is an m -dimensional Euclidean space (\mathbb{R}^m, g_0) . To summarize, the present paper derives another unified formulation of (1.9) as (2.6) for maps from (\mathbb{R}^m, g_0) or (\mathbb{T}^m, g_0) into locally hermitian symmetric spaces including $G_{n,k}$ and G_n^k . In addition, Theorem 1.1 concludes time-local existence of a generalized bi-Schrödinger flow for maps from (\mathbb{T}, g_0) into the compact locally hermitian symmetric spaces including $G_{n,k}$ and other examples which have not been studied so far. Finally, we stress that the assumption $\nabla R = 0$ works both for enjoying the solvable structure of (1.1)-(1.2) and for reducing (1.9) to the form (2.6).

The organization of the present paper is as follows: In Section 2, the equation for the generalized bi-Schrödinger flow is formulated as (2.6) or (2.7), and the relationship among (1.1), (1.3), and (1.4) is discussed. In Section 3, the proof of Theorem 1.1 is demonstrated.

2. THE GENERALIZED BI-SCHRÖDINGER FLOW AND THE RELATION WITH OTHER EQUATIONS

First, we formulate the equation (1.9) for maps from (\mathbb{T}^m, g_0) to locally hermitian symmetric spaces as (2.6) or (2.7), and discuss the correspondence to (1.1). Next, we discuss the relationship among (1.1), (1.3), and (1.4).

2.1. The formulation of the generalized bi-Schrödinger flow. In this subsection, suppose that $(M, g) = (\mathbb{T}^m, g_0)$ and (N, J, h) is a locally hermitian symmetric space. We formulate (1.9).

Before that, let us first recall the following basic properties of the Riemannian curvature tensor: For a map $u : \mathbb{T}^m \rightarrow N$ and for any $Y_1, \dots, Y_4 \in \Gamma(u^{-1}TN)$, it holds that

- (i) $R(Y_1, Y_2) = -R(Y_2, Y_1)$,
- (ii) $h(R(Y_1, Y_2)Y_3, Y_4) = h(R(Y_3, Y_4)Y_1, Y_2) = h(R(Y_4, Y_3)Y_2, Y_1)$,
- (iii) $R(Y_1, Y_2)Y_3 + R(Y_2, Y_3)Y_1 + R(Y_3, Y_1)Y_2 = 0$,
- (iv) $R(Y_1, Y_2)J_u Y_3 = J_u R(Y_1, Y_2)Y_3$,
- (v) $R(J_u Y_1, J_u Y_2)Y_3 = R(Y_1, Y_2)Y_3$,
- (vi) $R(J_u Y_1, Y_2)Y_3 = -R(Y_1, J_u Y_2)Y_3 = R(J_u Y_2, Y_1)Y_3$.

The property (iii) is called the Bianchi identity. The property (iv) holds since (N, J, h) is a Kähler manifold. The property (v) follows from (ii), (iv), and the invariance of h under the action J_u . The property (vi) follows from (i), (v), and $J_u^2 = -I_d$.

In addition, the condition $\nabla R = 0$ imposed on (N, h) implies

$$\begin{aligned} \nabla_x \{R(Y_1, Y_2)Y_3\} &= (\nabla_x R)(Y_1, Y_2)Y_3 + R(\nabla_x Y_1, Y_2)Y_3 + R(Y_1, \nabla_x Y_2)Y_3 + R(Y_1, Y_2)\nabla_x Y_3 \\ &= R(\nabla_x Y_1, Y_2)Y_3 + R(Y_1, \nabla_x Y_2)Y_3 + R(Y_1, Y_2)\nabla_x Y_3. \end{aligned} \quad (2.1)$$

These properties (i)-(vi) and (2.1) are applied not only in this section but also in the proof of Theorem 1.1.

Let us next formulate three energy functionals stated in Introduction: For a map $u : \mathbb{T}^m \rightarrow N$, $E(u)$ and $E_2(u)$ is respectively formulated as follows:

$$E(u) = \frac{1}{2} \sum_{k=1}^m \int_{\mathbb{T}^m} |u_{x_k}|_h^2 dx \quad \text{and} \quad E_2(u) = \frac{1}{2} \int_{\mathbb{T}^m} \left| \sum_{k=1}^m \nabla_{x_k} u_{x_k} \right|_h^2 dx,$$

where $|\cdot|_h = \{h(\cdot, \cdot)\}^{1/2}$. Moreover, since the right hand side of $E_*(u)$ is independent of the choice of $\{e_1, \dots, e_m\}$, we set $e_k = \partial/\partial x_k$ ($k = 1, 2, \dots, m$) to deduce

$$\begin{aligned} E_*(u) &= \int_{\mathbb{T}^m} \sum_{i,j,k,\ell=1}^m \delta_{ij} \delta_{k\ell} h(R(\nabla_{e_i} u, J_u \nabla_{e_j} u) J_u \nabla_{e_k} u, \nabla_{e_\ell} u) dx \\ &= \int_{\mathbb{T}^m} \sum_{i,k=1}^m h(R(\nabla_{e_i} u, J_u \nabla_{e_i} u) J_u \nabla_{e_k} u, \nabla_{e_k} u) dx \\ &= \int_{\mathbb{T}^m} \sum_{i,k=1}^m h(R(u_{x_i}, J_u u_{x_i}) J_u u_{x_k}, u_{x_k}) dx. \end{aligned}$$

Having them in mind, we are now ready to compute the gradient flow

$$\nabla E_{\alpha,\beta,\gamma}(u) = \alpha \nabla E(u) + \beta \nabla E_2(u) + \gamma \nabla E_*(u).$$

We demonstrate the computation of $\nabla E_*(u)$ below. First, following the method in [13], we construct the variation of $u : \mathbb{T}^m \rightarrow N$ with given initial velocity $\xi \in \Gamma(u^{-1}TN)$ by $U : \mathbb{T}^m \times \mathbb{R} \rightarrow N$, where $U(x, \varepsilon) = \exp_{u(x)} [\varepsilon \xi(x)]$ and $\exp_{u(x)} : T_{u(x)}N \rightarrow N$ is the exponential map at $u(x) \in N$. Next, since $\nabla E_*(u)$ is given by

$$\left. \frac{d}{d\varepsilon} E_*(U) \right|_{\varepsilon=0} = \int_{\mathbb{T}^m} h(\nabla E_*(u), \xi) dx, \quad (2.2)$$

we compute the left hand side of (2.2), where $\nabla R = 0$ is used. Indeed, by using (ii) and (2.1) with $u = U$, we see

$$\begin{aligned} \frac{d}{d\varepsilon} E_*(U) &= \sum_{i,k=1}^m \int_{\mathbb{T}^m} h(R(\nabla_\varepsilon U_{x_i}, J_U U_{x_i}) J_U U_{x_k}, U_{x_k}) dx \\ &\quad + \sum_{i,k=1}^m \int_{\mathbb{T}^m} h(R(U_{x_i}, \nabla_\varepsilon J_U U_{x_i}) J_U U_{x_k}, U_{x_k}) dx \\ &\quad + \sum_{i,k=1}^m \int_{\mathbb{T}^m} h(R(U_{x_i}, J_U U_{x_i}) \nabla_\varepsilon J_U U_{x_k}, U_{x_k}) dx \end{aligned}$$

$$\begin{aligned}
& + \sum_{i,k=1}^m \int_{\mathbb{T}^m} h(R(U_{x_i}, J_U U_{x_i}) J_U U_{x_k}, \nabla_\varepsilon U_{x_k}) dx \\
& = 2 \sum_{i,k=1}^m \int_{\mathbb{T}^m} h(R(\nabla_\varepsilon U_{x_i}, J_U U_{x_i}) J_U U_{x_k}, U_{x_k}) dx \\
& \quad + 2 \sum_{i,k=1}^m \int_{\mathbb{T}^m} h(R(U_{x_i}, \nabla_\varepsilon J_U U_{x_i}) J_U U_{x_k}, U_{x_k}) dx.
\end{aligned}$$

Since (N, J, h) is a Kähler manifold, $\nabla_\varepsilon J_U = J_U \nabla_\varepsilon$ holds. Using this and (vi) with $u = U$, we have

$$\begin{aligned}
h(R(U_{x_i}, \nabla_\varepsilon J_U U_{x_i}) J_U U_{x_k}, U_{x_k}) & = h(R(U_{x_i}, J_U \nabla_\varepsilon U_{x_i}) J_U U_{x_k}, U_{x_k}) \\
& = h(R(\nabla_\varepsilon U_{x_i}, J_U U_{x_i}) J_U U_{x_k}, U_{x_k}),
\end{aligned}$$

which implies

$$\frac{d}{d\varepsilon} E_\star(U) = 4 \sum_{i,k=1}^m \int_{\mathbb{T}^m} h(R(\nabla_\varepsilon U_{x_i}, J_U U_{x_i}) J_U U_{x_k}, U_{x_k}) dx.$$

Furthermore, by integrating by parts and by using $\nabla_\varepsilon U_{x_i} = \nabla_{x_i} U_\varepsilon$, (i), (ii) and (vi) with $u = U$, we obtain

$$\begin{aligned}
\frac{d}{d\varepsilon} E_\star(U) & = 4 \sum_{i,k=1}^m \int_{\mathbb{T}^m} h(R(\nabla_{x_i} U_\varepsilon, J_U U_{x_i}) J_U U_{x_k}, U_{x_k}) dx \\
& = -4 \sum_{i,k=1}^m \int_{\mathbb{T}^m} h(R(U_\varepsilon, J_U \nabla_{x_i} U_{x_i}) J_U U_{x_k}, U_{x_k}) dx \\
& \quad - 4 \sum_{i,k=1}^m \int_{\mathbb{T}^m} h(R(U_\varepsilon, J_U U_{x_i}) J_U \nabla_{x_i} U_{x_k}, U_{x_k}) dx \\
& \quad - 4 \sum_{i,k=1}^m \int_{\mathbb{T}^m} h(R(U_\varepsilon, J_U U_{x_i}) J_U U_{x_k}, \nabla_{x_i} U_{x_k}) dx \\
& = -4 \sum_{i,k=1}^m \int_{\mathbb{T}^m} h(R(U_\varepsilon, J_U \nabla_{x_i} U_{x_i}) J_U U_{x_k}, U_{x_k}) dx \\
& \quad + 8 \sum_{i,k=1}^m \int_{\mathbb{T}^m} h(R(U_\varepsilon, J_U U_{x_i}) \nabla_{x_i} U_{x_k}, J_U U_{x_k}) dx.
\end{aligned}$$

Note that $U_\varepsilon(x, \varepsilon)|_{\varepsilon=0} = \xi(x)$ and $U(x, \varepsilon)|_{\varepsilon=0} = u(x)$ follow from the definition of the map U . Therefore, by letting $\varepsilon \rightarrow 0$ and by using (ii), we get

$$\begin{aligned}
\left. \frac{d}{d\varepsilon} E_\star(U) \right|_{\varepsilon=0} & = -4 \sum_{i,k=1}^m \int_{\mathbb{T}^m} h(R(\xi, J_u \nabla_{x_i} u_{x_i}) J_u u_{x_k}, u_{x_k}) dx \\
& \quad + 8 \sum_{i,k=1}^m \int_{\mathbb{T}^m} h(R(\xi, J_u u_{x_i}) \nabla_{x_i} u_{x_k}, J_u u_{x_k}) dx \\
& = -4 \sum_{i,k=1}^m \int_{\mathbb{T}^m} h(\xi, R(u_{x_k}, J_u u_{x_k}) J_u \nabla_{x_i} u_{x_i}) dx
\end{aligned}$$

$$+ 8 \sum_{i,k=1}^m \int_{\mathbb{T}^m} h(\xi, R(J_u u_{x_k}, \nabla_{x_i} u_{x_k}) J_u u_{x_i}) dx.$$

Moreover, noting h is invariant under J_u and $J_u^2 = -I_d$, we use (iii) and (iv) to deduce

$$\begin{aligned} & h(\xi, R(J_u u_{x_k}, \nabla_{x_i} u_{x_k}) J_u u_{x_i}) \\ &= -h(J_u \xi, R(J_u u_{x_k}, \nabla_{x_i} u_{x_k}) u_{x_i}) \quad (\because \text{(iv)}) \\ &= h(J_u \xi, R(\nabla_{x_i} u_{x_k}, u_{x_i}) J_u u_{x_k}) + h(J_u \xi, R(u_{x_i}, J_u u_{x_k}) \nabla_{x_i} u_{x_k}) \quad (\because \text{(iii)}) \\ &= h(\xi, R(\nabla_{x_i} u_{x_k}, u_{x_i}) u_{x_k}) - h(\xi, R(u_{x_i}, J_u u_{x_k}) J_u \nabla_{x_i} u_{x_k}). \quad (\because \text{(iv)}) \end{aligned}$$

Substituting this, we obtain

$$\begin{aligned} \left. \frac{d}{d\varepsilon} E_\star(U) \right|_{\varepsilon=0} &= -4 \sum_{i,k=1}^m \int_{\mathbb{T}^m} h(\xi, R(u_{x_k}, J_u u_{x_k}) J_u \nabla_{x_i} u_{x_i}) dx \\ &\quad + 8 \sum_{i,k=1}^m \int_{\mathbb{T}^m} h(\xi, R(\nabla_{x_i} u_{x_k}, u_{x_i}) u_{x_k}) dx \\ &\quad - 8 \sum_{i,k=1}^m \int_{\mathbb{T}^m} h(\xi, R(u_{x_i}, J_u u_{x_k}) J_u \nabla_{x_i} u_{x_k}) dx. \end{aligned}$$

Thus, comparing with (2.2), we obtain

$$\begin{aligned} \nabla E_\star(u) &= -4 \sum_{i,k=1}^m R(u_{x_k}, J_u u_{x_k}) J_u \nabla_{x_i} u_{x_i} + 8 \sum_{i,k=1}^m R(\nabla_{x_i} u_{x_k}, u_{x_i}) u_{x_k} \\ &\quad - 8 \sum_{i,k=1}^m R(u_{x_i}, J_u u_{x_k}) J_u \nabla_{x_i} u_{x_k}. \end{aligned} \quad (2.3)$$

In the same way, we can compute to see

$$\nabla E(u) = - \sum_{k=1}^m \nabla_{x_k} u_{x_k}, \quad (2.4)$$

$$\nabla E_2(u) = \sum_{k,\ell=1}^m \{ \nabla_{x_k}^2 \nabla_{x_\ell} u_{x_\ell} + R(\nabla_{x_\ell} u_{x_\ell}, u_{x_k}) u_{x_k} \}. \quad (2.5)$$

We omit the detail, since (2.4) and (2.5) are already well-known. (See, e.g., [13, Section 2.2] for (2.5).) Indeed, the form of the right hand side of (2.4) and (2.5) actually agrees with that of the harmonic map equation and the bi-harmonic map equation respectively.

Combining (2.3), (2.4), and (2.5), we obtain

$$\begin{aligned} \nabla E_{\alpha,\beta,\gamma}(u) &= \beta \sum_{k,\ell=1}^m \nabla_{x_k}^2 \nabla_{x_\ell} u_{x_\ell} - \alpha \sum_{k=1}^m \nabla_{x_k} u_{x_k} \\ &\quad + \beta \sum_{k,\ell=1}^m R(\nabla_{x_\ell} u_{x_\ell}, u_{x_k}) u_{x_k} - 4\gamma \sum_{i,k=1}^m R(u_{x_k}, J_u u_{x_k}) J_u \nabla_{x_i} u_{x_i} \\ &\quad + 8\gamma \sum_{i,k=1}^m R(\nabla_{x_i} u_{x_k}, u_{x_i}) u_{x_k} - 8\gamma \sum_{i,k=1}^m R(u_{x_i}, J_u u_{x_k}) J_u \nabla_{x_i} u_{x_k}. \end{aligned}$$

Therefore, by using (iv) and (vi), we see that the partial differential equation for the generalized bi-Schrödinger flow is governed by

$$\begin{aligned}
u_t &= \beta J_u \sum_{k,\ell=1}^m \nabla_{x_k}^2 \nabla_{x_\ell} u_{x_\ell} - \alpha J_u \sum_{k=1}^m \nabla_{x_k} u_{x_k} \\
&\quad + \beta \sum_{k,\ell=1}^m R(\nabla_{x_\ell} u_{x_\ell}, u_{x_k}) J_u u_{x_k} + 4\gamma \sum_{i,k=1}^m R(u_{x_k}, J_u u_{x_k}) \nabla_{x_i} u_{x_i} \\
&\quad + 8\gamma \sum_{i,k=1}^m R(\nabla_{x_i} u_{x_k}, u_{x_i}) J_u u_{x_k} + 8\gamma \sum_{i,k=1}^m R(u_{x_i}, J_u u_{x_k}) \nabla_{x_i} u_{x_k} \\
&= \beta J_u \sum_{k,\ell=1}^m \nabla_{x_k}^2 \nabla_{x_\ell} u_{x_\ell} - \alpha J_u \sum_{k=1}^m \nabla_{x_k} u_{x_k} \\
&\quad + \beta \sum_{k,\ell=1}^m R(\nabla_{x_\ell} u_{x_\ell}, u_{x_k}) J_u u_{x_k} - 4\gamma \sum_{i,k=1}^m R(J_u u_{x_k}, u_{x_k}) \nabla_{x_i} u_{x_i} \\
&\quad + 8\gamma \sum_{i,k=1}^m R(\nabla_{x_i} u_{x_k}, u_{x_i}) J_u u_{x_k} - 8\gamma \sum_{i,k=1}^m R(J_u u_{x_i}, u_{x_k}) \nabla_{x_i} u_{x_k}. \tag{2.6}
\end{aligned}$$

Specifically if $m = 1$, it is obvious that the equation (2.6) reads

$$\begin{aligned}
u_t &= \beta J_u \nabla_x^3 u_x - \alpha J_u \nabla_x u_x \\
&\quad + (\beta + 8\gamma) R(\nabla_x u_x, u_x) J_u u_x - 12\gamma R(J_u u_x, u_x) \nabla_x u_x, \tag{2.7}
\end{aligned}$$

which is just (1.1) where $a = \beta$, $\lambda = -\alpha$, $b = \beta + 8\gamma$, $c = -12\gamma$.

2.2. The relationship among (1.1), (1.3), and (1.4). Suppose that (N, J, h) is a Riemann surface with constant sectional curvature S . Let u be a smooth solution to (1.1). By the definition of the sectional curvature,

$$R(X, Y)Z = S \{h(Y, Z)X - h(X, Z)Y\}$$

holds for any $X, Y, Z \in \Gamma(u^{-1}TN)$. Using this, (iv), and $J_u^2 = -I_d$, we have

$$\begin{aligned}
R(\nabla_x u_x, u_x) J_u u_x &= J_u R(\nabla_x u_x, u_x) u_x \\
&= S h(u_x, u_x) J_u \nabla_x u_x - S h(\nabla_x u_x, u_x) J_u u_x, \tag{2.8}
\end{aligned}$$

$$\begin{aligned}
R(J_u u_x, u_x) \nabla_x u_x &= S \{h(u_x, \nabla_x u_x) J_u u_x - h(J_u u_x, \nabla_x u_x) u_x\} \\
&= S h(\nabla_x u_x, u_x) J_u u_x + S J_u h(\nabla_x u_x, J_u u_x) J_u u_x. \tag{2.9}
\end{aligned}$$

Since N is here a two-dimensional real manifold, $\frac{u_x(t, x)}{|u_x(t, x)|_h}$ and $\frac{J_u(t, x) u_x(t, x)}{|u_x(t, x)|_h}$ form a basis for $T_{u(t, x)}N$ if $u_x(t, x) \neq 0$. Therefore, we notice that

$$h(u_x, u_x)Y = h(Y, u_x)u_x + h(Y, J_u u_x)J_u u_x \tag{2.10}$$

holds for any $Y \in \Gamma(u^{-1}TN)$. Using (2.10) with $Y = \nabla_x u_x$, we have

$$\begin{aligned}
J_u h(\nabla_x u_x, J_u u_x) J_u u_x &= J_u \{h(u_x, u_x) \nabla_x u_x - h(\nabla_x u_x, u_x) u_x\} \\
&= h(u_x, u_x) J_u \nabla_x u_x - h(\nabla_x u_x, u_x) J_u u_x. \tag{2.11}
\end{aligned}$$

Substituting (2.11) into (2.9), we have

$$R(J_u u_x, u_x) \nabla_x u_x = S h(u_x, u_x) J_u \nabla_x u_x. \tag{2.12}$$

Collecting (2.8) and (2.12), we see that the equation (1.1) reads

$$u_t = a J_u \nabla_x^3 u_x + \lambda J_u \nabla_x u_x + (b + c) S h(u_x, u_x) J_u \nabla_x u_x - b S h(\nabla_x u_x, u_x) J_u u_x,$$

which is nothing but (1.4) where $b_1 = a$, $b_2 = \lambda$, $b_3 = (b + c)S$, and $b_4 = -bS$.

Furthermore, it follows from [19] that the \mathbb{S}^2 -valued model (1.3) is reformulated as (1.4) where N is the canonical two-sphere \mathbb{S}^2 with $S = 1$ and $b_1 = a_1$, $b_2 = 1$, $b_3 = a_2 - a_1$, $b_4 = -5a_1 + 2a_2$. In other words, (1.3) is reformulated as (1.1) where $N = \mathbb{S}^2$, $a = a_1$, $\lambda = 1$, $b = 5a_1 - 2a_2$, $c = -6a_1 + 3a_2$.

3. LOCAL EXISTENCE

The purpose of this section is to prove Theorem 1.1.

Proof of Theorem 1.1. Suppose $k \geq 4$. It suffices to construct a solution in the positive time direction. We start with the case where $u_0 \in C^\infty(\mathbb{T}; N)$. We can construct a family of parabolic regularized approximating solutions by solving

$$u_t = (-\varepsilon + a J_u) \nabla_x^3 u_x + b R(\nabla_x u_x, u_x) J_u u_x + c R(J_u u_x, u_x) \nabla_x u_x + \lambda J_u \nabla_x u_x \quad \text{in } (0, \infty) \times \mathbb{T}, \quad (3.1)$$

$$u(0, x) = u_0(x) \quad \text{in } \mathbb{T} \quad (3.2)$$

for each fixed $\varepsilon \in (0, 1]$. Since the parabolic term $-\varepsilon \nabla_x^3 u_x$ is added, we can handle (3.1) as a fourth-order quasilinear parabolic system. In fact, we can show the following:

Lemma 3.1. *For each $\varepsilon \in (0, 1]$, there exists a positive constant T_ε depending on ε and $\|u_{0x}\|_{H^4(\mathbb{T}; TN)}$ such that (3.1)-(3.2) possesses a unique solution $u \in C^\infty([0, T_\varepsilon] \times \mathbb{T}; N)$.*

Lemma 3.1 can be easily proved by following the argument in [5, Lemma 3.1] or [22, Lemma 2.2]. The argument is based on the mix of a sixth-order parabolic regularization and geometric classical energy method. In what follows, we denote by u^ε the solution to (3.1)-(3.2) in Lemma 3.1. This presents the family $\{u^\varepsilon\}_{\varepsilon \in (0, 1]}$.

We next obtain a uniform lower bound T of $\{T_\varepsilon\}_{\varepsilon \in (0, 1]}$ and show that $\{u^\varepsilon\}_{\varepsilon \in (0, 1]}$ is bounded in $L^\infty(0, T; H^k(\mathbb{T}; TN))$. The classical energy estimate for $\|u^\varepsilon\|_{H^k(\mathbb{T}; TN)}$ is found to cause loss of derivatives. To eliminate the loss of derivatives, we introduce a gauge transformed function V_k^ε defined by

$$V_k^\varepsilon = \nabla_x^k u_x^\varepsilon + \Lambda^\varepsilon, \quad (3.3)$$

where $\Lambda^\varepsilon = \Lambda_1^\varepsilon + \Lambda_2^\varepsilon$ with

$$\begin{aligned} \Lambda_1^\varepsilon &:= -\frac{e_1}{2a} R(\nabla_x^{k-2} u_x^\varepsilon, u_x^\varepsilon) u_x^\varepsilon = -\frac{e_1}{2a} \Phi_1 \nabla_x^{-2} (\nabla_x^k u_x^\varepsilon), \\ \Lambda_2^\varepsilon &:= \frac{e_2}{8a} R(J_{u^\varepsilon} u_x^\varepsilon, u_x^\varepsilon) J_{u^\varepsilon} \nabla_x^{k-2} u_x^\varepsilon = \frac{e_2}{8a} \Phi_2 \nabla_x^{-2} (\nabla_x^k u_x^\varepsilon). \end{aligned}$$

Here Φ_1 and Φ_2 are defined in Introduction and $e_1, e_2 \in \mathbb{R}$ are real constants to be decided later. Instead of the energy estimate for $\|u^\varepsilon\|_{H^k(\mathbb{T}; TN)}$, we consider the estimate for the energy $N_k(u^\varepsilon)$ defined by

$$N_k(u^\varepsilon(t)) = \sqrt{\|u_x^\varepsilon(t)\|_{H^{k-1}(\mathbb{T}; TN)}^2 + \|V_k^\varepsilon(t)\|_{L^2(\mathbb{T}; TN)}^2} \quad (3.4)$$

for $t \in [0, T^\varepsilon]$. Before the estimate, we restrict the time interval on $[0, T_\varepsilon^*]$ with T_ε^* defined by

$$T_\varepsilon^* = \sup \{T > 0 \mid N_4(u^\varepsilon(t)) \leq 2N_4(u_0) \text{ for all } t \in [0, T]\}.$$

The restriction of the time-interval and the Sobolev embedding ensure the existence of an ε -independent constant $C = C(\|u_{0x}\|_{H^4(\mathbb{T};TN)}) > 1$ such that

$$\frac{1}{C} N_k(u^\varepsilon(t)) \leq \|u_x^\varepsilon(t)\|_{H^k(\mathbb{T};TN)} \leq C N_k(u^\varepsilon(t)) \quad \text{for any } t \in [0, T_\varepsilon^*]. \quad (3.5)$$

The equivalence of $N_k(u^\varepsilon)$ and $\|u_x^\varepsilon\|_{H^k(\mathbb{T};TN)}$ on $[0, T_\varepsilon^*]$ will be used frequently below. We shall show that there exists $T = T(\|u_{0x}\|_{H^4(\mathbb{T};TN)}) > 0$ which is independent of $\varepsilon \in (0, 1]$ and k such that $T_\varepsilon^* \geq T$ uniformly in $\varepsilon \in (0, 1]$ and that $\{N_k(u^\varepsilon)\}_{\varepsilon \in (0,1]}$ is bounded in $L^\infty(0, T)$. If it is true, this together with (3.5) implies that $\{u_x^\varepsilon\}_{\varepsilon \in (0,1]}$ is bounded in $L^\infty(0, T; H^k(\mathbb{T}; TN))$.

Having them in mind, we move on to the estimate for $N_k(u^\varepsilon)$. We set $u = u^\varepsilon$, $V_k = V_k^\varepsilon$, $\Lambda = \Lambda^\varepsilon$, $\Lambda_1 = \Lambda_1^\varepsilon$, $\Lambda_2 = \Lambda_2^\varepsilon$, $\|\cdot\|_{H^0(\mathbb{T};TN)} = \|\cdot\|_{L^2(\mathbb{T};TN)} = \|\cdot\|_{L^2}$, $\|\cdot\|_{H^m(\mathbb{T};TN)} = \|\cdot\|_{H^m}$ for $m = 1, \dots, k$, and $|\cdot|_h = \{h(\cdot, \cdot)\}^{1/2}$ for ease of notation. Since h is a hermitian metric, $h(J_u Y_1, J_u Y_2) = h(Y_1, Y_2)$ holds for any $Y_1, Y_2 \in \Gamma(u^{-1}TN)$. Since (N, J, h) is a Kähler manifold, $\nabla_x J_u = J_u \nabla_x$ and $\nabla_t J_u = J_u \nabla_t$ hold. Any positive constant which depends on $a, b, c, \lambda, k, \|u_{0x}\|_{H^4}$ and not on $\varepsilon \in (0, 1]$ will be denoted by the same C . Note that $k \geq 4$ and the Sobolev embedding $H^1(\mathbb{T}) \subset C(\mathbb{T})$ yield $\|\nabla_x^4 u_x\|_{L^\infty(0, T_\varepsilon^*; L^2)} \leq C$ and $\|\nabla_x^m u_x\|_{L^\infty((0, T_\varepsilon^*) \times \mathbb{T})} \leq C$ for $m = 0, 1, \dots, 3$. These properties will be used without any comment in this section.

We now investigate the energy estimate for $\|V_k\|_{L^2}^2$. The starting point is the observation that

$$\frac{1}{2} \frac{d}{dt} \|V_k\|_{L^2}^2 = \int_{\mathbb{T}} h(\nabla_t V_k, V_k) dx. \quad (3.6)$$

To evaluate the right hand side (denoted by RHS hereafter), we compute $\nabla_t V_k = \nabla_t \nabla_x^k u_x + \nabla_t \Lambda$ below.

We begin with the computation of $\nabla_t \nabla_x^k u_x$. Recalling that $\nabla_x u_t = \nabla_t u_x$ and $(\nabla_x \nabla_t - \nabla_t \nabla_x)Y = R(u_x, u_t)Y$ for any $Y \in \Gamma(u^{-1}TN)$, we have

$$\nabla_t \nabla_x^k u_x = \nabla_x^{k+1} u_t + \sum_{m=0}^{k-1} \nabla_x^{k-1-m} \{R(u_t, u_x) \nabla_x^m u_x\} =: \nabla_x^{k+1} u_t + Q. \quad (3.7)$$

We compute the RHS of (3.7) using (3.1). We first look at the first term of the RHS of (3.7). A simple computation shows

$$\nabla_x^{k+1} u_t = -\varepsilon \nabla_x^4 (\nabla_x^k u_x) + a J_u \nabla_x^4 (\nabla_x^k u_x) + \lambda J_u \nabla_x^2 (\nabla_x^k u_x) + b Q_{1,1} + c Q_{1,2}, \quad (3.8)$$

where $Q_{1,1} = \nabla_x^{k+1} \{R(\nabla_x u_x, u_x) J_u u_x\}$ and $Q_{1,2} = \nabla_x^{k+1} \{R(J_u u_x, u_x) \nabla_x u_x\}$. In the computation of $Q_{1,1}$ and $Q_{1,2}$, the assumption $\nabla R = 0$ is used. Indeed, by using (2.1) in the previous section and the product formula for covariant differentiation, we deduce

$$\begin{aligned} Q_{1,1} &= \sum_{\substack{q+r+s=k+1, \\ q, r, s \geq 0}} \frac{(k+1)!}{q!r!s!} R(\nabla_x^{q+1} u_x, \nabla_x^r u_x) J_u \nabla_x^s u_x \\ &= R(\nabla_x^{k+2} u_x, u_x) J_u u_x + (k+1) R(\nabla_x^{k+1} u_x, \nabla_x u_x) J_u u_x \\ &\quad + (k+1) R(\nabla_x^{k+1} u_x, u_x) J_u \nabla_x u_x + R(\nabla_x u_x, \nabla_x^{k+1} u_x) J_u u_x \\ &\quad + R(\nabla_x u_x, u_x) J_u \nabla_x^{k+1} u_x \\ &\quad + \sum_{\substack{q+r+s=k+1, \\ 0 \leq q \leq k-1, 0 \leq r, s \leq k}} \frac{(k+1)!}{q!r!s!} R(\nabla_x^{q+1} u_x, \nabla_x^r u_x) J_u \nabla_x^s u_x. \end{aligned}$$

Furthermore, the Sobolev embedding and the Gagliardo-Nirenberg inequality imply

$$Q_{1,1} = R(\nabla_x^{k+2} u_x, u_x) J_u u_x + (k+1) R(\nabla_x^{k+1} u_x, \nabla_x u_x) J_u u_x$$

$$\begin{aligned}
& + (k+1)R(\nabla_x^{k+1}u_x, u_x)J_u \nabla_x u_x + R(\nabla_x u_x, \nabla_x^{k+1}u_x)J_u u_x \\
& + R(\nabla_x u_x, u_x)J_u \nabla_x^{k+1}u_x + \mathcal{O}\left(\sum_{m=0}^k |\nabla_x^m u_x|_h\right).
\end{aligned} \tag{3.9}$$

In the same way as we compute $Q_{1,1}$, we compute $Q_{1,2}$ to obtain

$$\begin{aligned}
Q_{1,2} &= \sum_{\substack{q+r+s=k+1, \\ q,r,s \geq 0}} \frac{(k+1)!}{q!r!s!} R(J_u \nabla_x^q u_x, \nabla_x^r u_x) \nabla_x^{s+1} u_x \\
&= R(J_u u_x, u_x) \nabla_x^{k+2} u_x + (k+1)R(J_u \nabla_x u_x, u_x) \nabla_x^{k+1} u_x \\
&\quad + (k+1)R(J_u u_x, \nabla_x u_x) \nabla_x^{k+1} u_x + R(J_u \nabla_x^{k+1} u_x, u_x) \nabla_x u_x \\
&\quad + R(J_u u_x, \nabla_x^{k+1} u_x) \nabla_x u_x \\
&\quad + \sum_{\substack{q+r+s=k+1, \\ 0 \leq s \leq k-1, 0 \leq q, r \leq k}} \frac{(k+1)!}{q!r!s!} R(J_u \nabla_x^q u_x, \nabla_x^r u_x) \nabla_x^{s+1} u_x \\
&= \nabla_x \{R(J_u u_x, u_x) \nabla_x^{k+1} u_x\} + k R(J_u \nabla_x u_x, u_x) \nabla_x^{k+1} u_x \\
&\quad + k R(J_u u_x, \nabla_x u_x) \nabla_x^{k+1} u_x + R(J_u \nabla_x^{k+1} u_x, u_x) \nabla_x u_x \\
&\quad + R(J_u u_x, \nabla_x^{k+1} u_x) \nabla_x u_x + \mathcal{O}\left(\sum_{m=0}^k |\nabla_x^m u_x|_h\right).
\end{aligned} \tag{3.10}$$

We next look at Q which is the second term of the RHS of (3.7). By substituting (3.1),

$$\begin{aligned}
Q &= -\varepsilon \sum_{m=0}^{k-1} \nabla_x^{k-1-m} \{R(\nabla_x^3 u_x, u_x) \nabla_x^m u_x\} \\
&\quad + a \sum_{m=0}^{k-1} \nabla_x^{k-1-m} \{R(J_u \nabla_x^3 u_x, u_x) \nabla_x^m u_x\} \\
&\quad + \lambda \sum_{m=0}^{k-1} \nabla_x^{k-1-m} \{R(J_u \nabla_x u_x, u_x) \nabla_x^m u_x\} \\
&\quad + b \sum_{m=0}^{k-1} \nabla_x^{k-1-m} \{R(R(\nabla_x u_x, u_x) J_u u_x, u_x) \nabla_x^m u_x\} \\
&\quad + c \sum_{m=0}^{k-1} \nabla_x^{k-1-m} \{R(R(J_u u_x, u_x) \nabla_x u_x, u_x) \nabla_x^m u_x\}.
\end{aligned}$$

Thus, by using the Sobolev embedding and the Gagliardo-Nirenberg inequality, we obtain

$$Q = \varepsilon \mathcal{O}(|\nabla_x^{k+2} u_x|_h + |\nabla_x^{k+1} u_x|_h) + a Q_0 + \mathcal{O}\left(\sum_{m=0}^k |\nabla_x^m u_x|_h\right),$$

where

$$Q_0 = \sum_{m=0}^{k-1} \nabla_x^{k-1-m} \{R(J_u \nabla_x^3 u_x, u_x) \nabla_x^m u_x\}.$$

In the same way as we compute $Q_{1,1}$ and $Q_{1,2}$, we use the product formula for covariant differentiation to deduce

$$\begin{aligned}
Q_0 &= \nabla_x^{k-1} \{R(J_u \nabla_x^3 u_x, u_x)u_x\} + \nabla_x^{k-2} \{R(J_u \nabla_x^3 u_x, u_x)\nabla_x u_x\} \\
&\quad + \mathcal{O}\left(\sum_{m=0}^k |\nabla_x^m u_x|_h\right) \\
&= \sum_{\substack{q+r+s=k-1, \\ q,r,s \geq 0}} \frac{(k-1)!}{q!r!s!} R(J_u \nabla_x^{q+3} u_x, \nabla_x^r u_x) \nabla_x^s u_x \\
&\quad + R(J_u \nabla_x^{k+1} u_x, u_x) \nabla_x u_x + \mathcal{O}\left(\sum_{m=0}^k |\nabla_x^m u_x|_h\right) \\
&= R(J_u \nabla_x^{k+2} u_x, u_x)u_x + (k-1)R(J_u \nabla_x^{k+1} u_x, \nabla_x u_x)u_x \\
&\quad + k R(J_u \nabla_x^{k+1} u_x, u_x) \nabla_x u_x + \mathcal{O}\left(\sum_{m=0}^k |\nabla_x^m u_x|_h\right).
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
Q &= \varepsilon \mathcal{O}(|\nabla_x^{k+2} u_x|_h + |\nabla_x^{k+1} u_x|_h) + a R(J_u \nabla_x^{k+2} u_x, u_x)u_x \\
&\quad + a(k-1)R(J_u \nabla_x^{k+1} u_x, \nabla_x u_x)u_x + ak R(J_u \nabla_x^{k+1} u_x, u_x) \nabla_x u_x \\
&\quad + \mathcal{O}\left(\sum_{m=0}^k |\nabla_x^m u_x|_h\right). \tag{3.11}
\end{aligned}$$

By collecting (3.11) and (3.8) with (3.9) and with (3.10), we have

$$\begin{aligned}
\nabla_t \nabla_x^k u_x &= \{-\varepsilon \nabla_x^4 + a J_u \nabla_x^4 + \lambda J_u \nabla_x^2\} \nabla_x^k u_x + \varepsilon \mathcal{O}(|\nabla_x^{k+2} u_x|_h + |\nabla_x^{k+1} u_x|_h) \\
&\quad + a R(J_u \nabla_x^{k+2} u_x, u_x)u_x + b R(\nabla_x^{k+2} u_x, u_x)J_u u_x \\
&\quad + c \nabla_x \{R(J_u u_x, u_x) \nabla_x^{k+1} u_x\} + b R(\nabla_x u_x, u_x)J_u \nabla_x^{k+1} u_x \\
&\quad + a(k-1)R(J_u \nabla_x^{k+1} u_x, \nabla_x u_x)u_x + (ak+c) R(J_u \nabla_x^{k+1} u_x, u_x) \nabla_x u_x \\
&\quad + c R(J_u u_x, \nabla_x^{k+1} u_x) \nabla_x u_x + b(k+1)R(\nabla_x^{k+1} u_x, \nabla_x u_x)J_u u_x \\
&\quad + b R(\nabla_x u_x, \nabla_x^{k+1} u_x)J_u u_x + b(k+1)R(\nabla_x^{k+1} u_x, u_x)J_u \nabla_x u_x \\
&\quad + ck R(J_u \nabla_x u_x, u_x) \nabla_x^{k+1} u_x + ck R(J_u u_x, \nabla_x u_x) \nabla_x^{k+1} u_x \\
&\quad + \mathcal{O}\left(\sum_{m=0}^k |\nabla_x^m u_x|_h\right). \tag{3.12}
\end{aligned}$$

Here, using (i)-(vi) in the previous section, we can rewrite the each term of the RHS of (3.12) separately. The result of the computation is as follows:

$$\begin{aligned}
R(J_u \nabla_x^{k+2} u_x, u_x)u_x &= -R(\nabla_x^{k+2} u_x, J_u u_x)u_x, \quad (\because \text{(vi)}) \tag{3.13} \\
R(\nabla_x^{k+2} u_x, u_x)J_u u_x &= -R(u_x, J_u u_x) \nabla_x^{k+2} u_x - R(J_u u_x, \nabla_x^{k+2} u_x)u_x \quad (\because \text{(iii)}) \\
&= R(J_u u_x, u_x) \nabla_x^{k+2} u_x + R(\nabla_x^{k+2} u_x, J_u u_x)u_x \quad (\because \text{(i)}) \\
&= \nabla_x \{R(J_u u_x, u_x) \nabla_x^{k+1} u_x\} - R(J_u \nabla_x u_x, u_x) \nabla_x^{k+1} u_x \\
&\quad - R(J_u u_x, \nabla_x u_x) \nabla_x^{k+1} u_x + R(\nabla_x^{k+2} u_x, J_u u_x)u_x
\end{aligned}$$

$$= \nabla_x \{ R(J_u u_x, u_x) \nabla_x^{k+1} u_x \} - 2 R(J_u \nabla_x u_x, u_x) \nabla_x^{k+1} u_x \\ + R(\nabla_x^{k+2} u_x, J_u u_x) u_x, \quad (\because \text{(vi)}) \quad (3.14)$$

$$R(J_u \nabla_x^{k+1} u_x, u_x) \nabla_x u_x = R(J_u u_x, \nabla_x^{k+1} u_x) \nabla_x u_x \quad (\because \text{(vi)}) \quad (3.15)$$

$$= -R(\nabla_x^{k+1} u_x, \nabla_x u_x) J_u u_x - R(\nabla_x u_x, J_u u_x) \nabla_x^{k+1} u_x \quad (\because \text{(iii)}) \\ = -R(\nabla_x^{k+1} u_x, \nabla_x u_x) J_u u_x + R(J_u \nabla_x u_x, u_x) \nabla_x^{k+1} u_x, \quad (\because \text{(vi)}) \quad (3.16)$$

$$R(J_u \nabla_x^{k+1} u_x, \nabla_x u_x) u_x \\ = -R(\nabla_x u_x, u_x) J_u \nabla_x^{k+1} u_x - R(u_x, J_u \nabla_x^{k+1} u_x) \nabla_x u_x \quad (\because \text{(iii)}) \\ = -R(\nabla_x u_x, u_x) J_u \nabla_x^{k+1} u_x + R(J_u \nabla_x^{k+1} u_x, u_x) \nabla_x u_x \quad (\because \text{(i)}) \\ = -R(\nabla_x u_x, u_x) J_u \nabla_x^{k+1} u_x - R(\nabla_x^{k+1} u_x, \nabla_x u_x) J_u u_x \\ + R(J_u \nabla_x u_x, u_x) \nabla_x^{k+1} u_x, \quad (\because \text{(3.16)}) \quad (3.17)$$

$$R(\nabla_x u_x, \nabla_x^{k+1} u_x) J_u u_x = -R(\nabla_x^{k+1} u_x, \nabla_x u_x) J_u u_x, \quad (\because \text{(i)}) \quad (3.18)$$

$$R(J_u u_x, \nabla_x u_x) \nabla_x^{k+1} u_x = R(J_u \nabla_x u_x, u_x) \nabla_x^{k+1} u_x. \quad (\because \text{(vi)}) \quad (3.19)$$

Substituting (3.13)-(3.19) into (3.12), we obtain

$$\nabla_t \nabla_x^k u_x = \{ -\varepsilon \nabla_x^4 + a J_u \nabla_x^4 + \lambda J_u \nabla_x^2 \} \nabla_x^k u_x + \varepsilon \mathcal{O}(|\nabla_x^{k+2} u_x|_h + |\nabla_x^{k+1} u_x|_h) \\ + (-a + b) R(\nabla_x^{k+2} u_x, J_u u_x) u_x + (b + c) \nabla_x \{ R(J_u u_x, u_x) \nabla_x^{k+1} u_x \} \\ + c_1 R(J_u \nabla_x u_x, u_x) \nabla_x^{k+1} u_x + c_2 R(\nabla_x u_x, u_x) J_u \nabla_x^{k+1} u_x \\ + c_3 R(\nabla_x^{k+1} u_x, \nabla_x u_x) J_u u_x + c_4 R(\nabla_x^{k+1} u_x, u_x) J_u \nabla_x u_x \\ + \mathcal{O} \left(\sum_{m=0}^k |\nabla_x^m u_x|_h \right), \quad (3.20)$$

where

$$c_1 = -2b + a(k-1) + (ak+c) + c + 2ck = (2k-1)a - 2b + (2k+2)c, \quad (3.21)$$

$$c_2 = -(k-1)a + b, \quad (3.22)$$

$$c_3 = -a(k-1) - (ak+c) - c + b(k+1) - b = -(2k-1)a + kb - 2c, \quad (3.23)$$

$$c_4 = (k+1)b. \quad (3.24)$$

Furthermore, we rewrite $R(\nabla_x^{k+1} u_x, \nabla_x u_x) J_u u_x$ and $R(\nabla_x^{k+1} u_x, u_x) J_u \nabla_x u_x$ by using A_i ($i = 1, 2, 3$) defined for any $Y \in \Gamma(u^{-1}TN)$ by

$$A_1 Y = R(Y, \nabla_x u_x) J_u u_x - R(Y, u_x) J_u \nabla_x u_x,$$

$$A_2 Y = R(Y, \nabla_x u_x) J_u u_x + R(Y, J_u u_x) \nabla_x u_x,$$

$$A_3 Y = R(Y, u_x) J_u \nabla_x u_x + R(Y, J_u \nabla_x u_x) u_x.$$

Remark 3.2. It is to be emphasized that A_i ($i = 1, 2, 3$) is symmetric, that is,

$$h(A_i Y, Z) = h(Y, A_i Z) \quad (i = 1, 2, 3) \quad (3.25)$$

for any $Y, Z \in \Gamma(u^{-1}TN)$. If $i = 1$, (3.25) follows from

$$h(A_1 Y, Z) = h(R(Y, \nabla_x u_x) J_u u_x, Z) - h(R(Y, u_x) J_u \nabla_x u_x, Z) \\ = -h(R(\nabla_x u_x, J_u u_x) Y, Z) - h(R(J_u u_x, Y) \nabla_x u_x, Z) \\ + h(R(u_x, J_u \nabla_x u_x) Y, Z) + h(R(J_u \nabla_x u_x, Y) u_x, Z) \quad (\because \text{(iii)}) \\ = -h(R(J_u u_x, Y) \nabla_x u_x, Z) + h(R(J_u \nabla_x u_x, Y) u_x, Z) \quad (\because \text{(vi)})$$

$$\begin{aligned}
&= h(R(Z, \nabla_x u_x) J_u u_x, Y) - h(R(Z, u_x) J_u \nabla_x u_x, Y) \quad (\because (i), (ii)) \\
&= h(A_1 Z, Y).
\end{aligned}$$

If $i = 2$ or $i = 3$, (3.25) easily follows from (i) and (ii). The property (3.25) will be used later.

By using A_1, A_2, A_3 , we can write

$$\begin{aligned}
4 R(Y, \nabla_x u_x) J_u u_x &= 2A_1 Y + 2 \{R(Y, \nabla_x u_x) J_u u_x + R(Y, u_x) J_u \nabla_x u_x\} \\
&= 2A_1 Y + A_2 Y + A_3 Y \\
&\quad + R(Y, \nabla_x u_x) J_u u_x - R(Y, J_u u_x) \nabla_x u_x \\
&\quad + R(Y, u_x) J_u \nabla_x u_x - R(Y, J_u \nabla_x u_x) u_x.
\end{aligned} \tag{3.26}$$

Here, it follows that

$$\begin{aligned}
&R(Y, \nabla_x u_x) J_u u_x + R(Y, u_x) J_u \nabla_x u_x \\
&= -R(\nabla_x u_x, J_u u_x) Y - R(J_u u_x, Y) \nabla_x u_x \\
&\quad - R(u_x, J_u \nabla_x u_x) Y - R(J_u \nabla_x u_x, Y) u_x \quad (\because (iii)) \\
&= 2 R(J_u \nabla_x u_x, u_x) Y + R(Y, J_u u_x) \nabla_x u_x + R(Y, J_u \nabla_x u_x) u_x. \quad (\because (i), (vi))
\end{aligned} \tag{3.27}$$

Combining (3.26) and (3.27), we have

$$R(Y, \nabla_x u_x) J_u u_x = \frac{1}{2} R(J_u \nabla_x u_x, u_x) Y + \frac{1}{2} A_1 Y + \frac{1}{4} A_2 Y + \frac{1}{4} A_3 Y. \tag{3.28}$$

In the same way, since

$$4 R(Y, u_x) J_u \nabla_x u_x = -2A_1 Y + 2 \{R(Y, \nabla_x u_x) J_u u_x + R(Y, u_x) J_u \nabla_x u_x\},$$

we have

$$R(Y, u_x) J_u \nabla_x u_x = \frac{1}{2} R(J_u \nabla_x u_x, u_x) Y - \frac{1}{2} A_1 Y + \frac{1}{4} A_2 Y + \frac{1}{4} A_3 Y. \tag{3.29}$$

Applying (3.28) and (3.29) with $Y = \nabla_x^{k+1} u_x$ to the RHS of (3.20), we obtain

$$\begin{aligned}
\nabla_t \nabla_x^k u_x &= \{-\varepsilon \nabla_x^4 + a J_u \nabla_x^4 + \lambda J_u \nabla_x^2\} \nabla_x^k u_x + \varepsilon \mathcal{O}(|\nabla_x^{k+2} u_x|_h + |\nabla_x^{k+1} u_x|_h) \\
&\quad + d_1 R(\nabla_x^{k+2} u_x, J_u u_x) u_x + d_2 \nabla_x \{R(J_u u_x, u_x) \nabla_x^{k+1} u_x\} \\
&\quad + d_3 R(J_u \nabla_x u_x, u_x) \nabla_x^{k+1} u_x + d_4 R(\nabla_x u_x, u_x) J_u \nabla_x^{k+1} u_x \\
&\quad + (d_5 A_1 + d_6 A_2 + d_7 A_3) \nabla_x^{k+1} u_x + \mathcal{O}\left(\sum_{m=0}^k |\nabla_x^m u_x|_h\right),
\end{aligned} \tag{3.30}$$

where $d_1 = -a + b$, $d_2 = b + c$, $d_3 = c_1 + c_3/2 + c_4/2$, $d_4 = c_2$, $d_5 = (c_3 - c_4)/2$, $d_6 = d_7 = (c_3 + c_4)/4$, and c_1, \dots, c_4 are given in (3.21)-(3.24). For instance, d_3 is given by

$$\begin{aligned}
d_3 &= (2k-1)a - 2b + (2k+2)c + \frac{1}{2} \{-(2k-1)a + kb - 2c\} + \frac{1}{2}(k+1)b \\
&= \left(k - \frac{1}{2}\right) a + \left(k - \frac{3}{2}\right) b + (2k+1)c.
\end{aligned} \tag{3.31}$$

The explicit form of these constants is not required except for d_1 and d_3 . Furthermore, substituting $\nabla_x^k u_x = V_k - \Lambda$ into the RHS of (3.30) and noting $\Lambda = \mathcal{O}(|\nabla_x^{k-2} u_x|_h)$, we find

$$\begin{aligned}
\nabla_t \nabla_x^k u_x &= \varepsilon \nabla_x^4 \Lambda - a J_u \nabla_x^4 \Lambda \\
&\quad + \{-\varepsilon \nabla_x^4 + a J_u \nabla_x^4 + \lambda J_u \nabla_x^2\} V_k + \varepsilon \mathcal{O}(|\nabla_x^{k+2} u_x|_h + |\nabla_x^{k+1} u_x|_h) \\
&\quad + d_1 R(\nabla_x^2 V_k, J_u u_x) u_x + d_2 \nabla_x \{R(J_u u_x, u_x) \nabla_x V_k\}
\end{aligned}$$

$$\begin{aligned}
& + d_3 R(J_u \nabla_x u_x, u_x) \nabla_x V_k + d_4 R(\nabla_x u_x, u_x) J_u \nabla_x V_k \\
& + (d_5 A_1 + d_6 A_2 + d_7 A_3) \nabla_x V_k + \mathcal{O} \left(\sum_{m=0}^k |\nabla_x^m u_x|_h \right).
\end{aligned} \tag{3.32}$$

Here, a simple computation yields

$$\varepsilon \nabla_x^4 \Lambda = \varepsilon \mathcal{O}(|\nabla_x^{k+2} u_x|_h + |\nabla_x^{k+1} u_x|_h) + \mathcal{O} \left(\sum_{m=0}^k |\nabla_x^m u_x|_h \right)$$

and $\nabla_x^4 \Lambda = \nabla_x^4 \Phi \nabla_x^{-2} \nabla_x^k u_x$, where $\Phi = -(e_1/2a)\Phi_1 + (e_2/8a)\Phi_2$. Consequently, we derive

$$\begin{aligned}
\nabla_t \nabla_x^k u_x &= -a J_u \nabla_x^4 \Phi \nabla_x^{-2} \nabla_x^k u_x + \{ -\varepsilon \nabla_x^4 + a J_u \nabla_x^4 + \lambda J_u \nabla_x^2 \} V_k \\
&+ \varepsilon \mathcal{O}(|\nabla_x^{k+2} u_x|_h + |\nabla_x^{k+1} u_x|_h) \\
&+ d_1 R(\nabla_x^2 V_k, J_u u_x) u_x + d_2 \nabla_x \{ R(J_u u_x, u_x) \nabla_x V_k \} \\
&+ d_3 R(J_u \nabla_x u_x, u_x) \nabla_x V_k + d_4 R(\nabla_x u_x, u_x) J_u \nabla_x V_k \\
&+ (d_5 A_1 + d_6 A_2 + d_7 A_3) \nabla_x V_k + \mathcal{O} \left(\sum_{m=0}^k |\nabla_x^m u_x|_h \right).
\end{aligned} \tag{3.33}$$

Next, we compute $\nabla_t \Lambda$. Using the product formula and noting that $\nabla_t u_x = \nabla_x u_t = \mathcal{O} \left(\sum_{m=0}^4 |\nabla_x^m u_x|_h \right)$, we have

$$\begin{aligned}
\nabla_t \Lambda &= -\frac{e_1}{2a} R(\nabla_t \nabla_x^{k-2} u_x, u_x) u_x - \frac{e_1}{2a} R(\nabla_x^{k-2} u_x, \nabla_t u_x) u_x \\
&- \frac{e_1}{2a} R(\nabla_x^{k-2} u_x, u_x) \nabla_t u_x + \frac{e_2}{8a} R(J_u u_x, u_x) J_u \nabla_t \nabla_x^{k-2} u_x \\
&+ \frac{e_2}{8a} R(J_u \nabla_t u_x, u_x) J_u \nabla_x^{k-2} u_x + \frac{e_2}{8a} R(J_u u_x, \nabla_t u_x) J_u \nabla_x^{k-2} u_x \\
&= -\frac{e_1}{2a} R(\nabla_t \nabla_x^{k-2} u_x, u_x) u_x + \frac{e_2}{8a} R(J_u u_x, u_x) J_u \nabla_t \nabla_x^{k-2} u_x \\
&+ \mathcal{O} \left(|\nabla_x^{k-2} u_x|_h \sum_{m=0}^4 |\nabla_x^m u_x|_h \right).
\end{aligned} \tag{3.34}$$

By the same computation as that we obtain $\nabla_t \nabla_x^k u_x$, we find

$$\begin{aligned}
\nabla_t \nabla_x^{k-2} u_x &= -\varepsilon \nabla_x^4 (\nabla_x^{k-2} u_x) + a J_u \nabla_x^4 (\nabla_x^{k-2} u_x) + \mathcal{O} \left(\sum_{m=0}^k |\nabla_x^m u_x|_h \right) \\
&= \varepsilon \mathcal{O}(|\nabla_x^{k+2} u_x|_h) + a J_u \nabla_x^{k+2} u_x + \mathcal{O} \left(\sum_{m=0}^k |\nabla_x^m u_x|_h \right).
\end{aligned} \tag{3.35}$$

Substituting (3.35) into (3.34) and observing $a J_u \nabla_x^{k+2} u_x = \nabla_x^{-2} (a J_u \nabla_x^4) \nabla_x^k u_x$, we obtain

$$\begin{aligned}
\nabla_t \Lambda &= -\frac{e_1}{2a} R(a J_u \nabla_x^{k+2} u_x, u_x) u_x + \frac{e_2}{8a} R(J_u u_x, u_x) J_u a J_u \nabla_x^{k+2} u_x \\
&+ \varepsilon \mathcal{O}(|\nabla_x^{k+2} u_x|_h) + \mathcal{O} \left(\sum_{m=0}^k |\nabla_x^m u_x|_h + |\nabla_x^{k-2} u_x|_h \sum_{m=0}^4 |\nabla_x^m u_x|_h \right) \\
&= \Phi \nabla_x^{-2} (a J_u \nabla_x^4) \nabla_x^k u_x
\end{aligned}$$

$$+ \varepsilon \mathcal{O}(|\nabla_x^{k+2} u_x|_h) + \mathcal{O} \left(\sum_{m=0}^k |\nabla_x^m u_x|_h + |\nabla_x^{k-2} u_x|_h \sum_{m=0}^4 |\nabla_x^m u_x|_h \right). \quad (3.36)$$

Consequently, by combining (3.33) and (3.36), we derive

$$\begin{aligned} \nabla_t V_k = & -[a J_u \nabla_x^4, \Phi \nabla_x^{-2}] \nabla_x^k u_x + \{ -\varepsilon \nabla_x^4 + a J_u \nabla_x^4 + \lambda J_u \nabla_x^2 \} V_k \\ & + \varepsilon \mathcal{O}(|\nabla_x^{k+2} u_x|_h + |\nabla_x^{k+1} u_x|_h) \\ & + d_1 R(\nabla_x^2 V_k, J_u u_x) u_x + d_2 \nabla_x \{ R(J_u u_x, u_x) \nabla_x V_k \} \\ & + d_3 R(J_u \nabla_x u_x, u_x) \nabla_x V_k + d_4 R(\nabla_x u_x, u_x) J_u \nabla_x V_k \\ & + (d_5 A_1 + d_6 A_2 + d_7 A_3) \nabla_x V_k \\ & + \mathcal{O} \left(\sum_{m=0}^k |\nabla_x^m u_x|_h + |\nabla_x^{k-2} u_x|_h \sum_{m=0}^4 |\nabla_x^m u_x|_h \right), \end{aligned} \quad (3.37)$$

where the symbol $[\cdot, \cdot]$ denotes the commutator, that is,

$$[a J_u \nabla_x^4, \Phi \nabla_x^{-2}] \nabla_x^k u_x = a J_u \nabla_x^4 (\Phi \nabla_x^{-2} \nabla_x^k u_x) - \Phi \nabla_x^{-2} (a J_u \nabla_x^4 \nabla_x^k u_x),$$

which plays the crucial role in the proof and is to be computed below. We start with

$$[a J_u \nabla_x^4, \Phi \nabla_x^{-2}] = -\frac{e_1}{2} [J_u \nabla_x^4, \Phi_1 \nabla_x^{-2}] + \frac{e_2}{8} [J_u \nabla_x^4, \Phi_2 \nabla_x^{-2}]. \quad (3.38)$$

In what follows, we use ∇_x^{-2} and ∇_x^{-1} which does not make sense in general. Fortunately, however, this makes sense in our computation, because they always act on the image of ∇_x^2 . First, from the product formula and $J_u \nabla_x = \nabla_x J_u$, it follows that

$$\begin{aligned} [J_u \nabla_x^4, \Phi_1 \nabla_x^{-2}] &= J_u \Phi_1 \nabla_x^2 + 4J_u (\nabla_x \Phi_1) \nabla_x + 6J_u (\nabla_x^2 \Phi_1) \\ &\quad + 4J_u (\nabla_x^3 \Phi_1) \nabla_x^{-1} + J_u (\nabla_x^4 \Phi_1) \nabla_x^{-2} - \Phi_1 J_u \nabla_x^2 \\ &= -2\Phi_1 J_u \nabla_x^2 + (J_u \Phi_1 + \Phi_1 J_u) \nabla_x^2 + 4J_u (\nabla_x \Phi_1) \nabla_x \\ &\quad + 6J_u (\nabla_x^2 \Phi_1) + 4J_u (\nabla_x^3 \Phi_1) \nabla_x^{-1} + J_u (\nabla_x^4 \Phi_1) \nabla_x^{-2}. \end{aligned} \quad (3.39)$$

Here $(\nabla_x^\ell \Phi_1)$, $\ell = 1, \dots, 4$, are defined by $(\nabla_x \Phi_1)Y = \nabla_x(\Phi_1 Y) - \Phi_1(\nabla_x Y)$, and $(\nabla_x^\ell \Phi_1)Y = \nabla_x \{ (\nabla_x^{\ell-1} \Phi_1)Y \} - (\nabla_x^{\ell-1} \Phi_1) \nabla_x Y$, $\ell = 2, 3, 4$, for any $Y \in \Gamma(u^{-1}TN)$. Recalling the definition of Φ_1 and using the properties (i)-(vi), (3.14), (3.28), and (3.29), we deduce

$$-2\Phi_1 J_u \nabla_x^2 = -2R(J_u \nabla_x^2 \cdot, u_x) u_x = 2R(\nabla_x^2 \cdot, J_u u_x) u_x, \quad (\because \text{(vi)}) \quad (3.40)$$

$$\begin{aligned} (J_u \Phi_1 + \Phi_1 J_u) \nabla_x^2 &= J_u R(\nabla_x^2 \cdot, u_x) u_x + R(J_u \nabla_x^2 \cdot, u_x) u_x \\ &= R(\nabla_x^2 \cdot, u_x) J_u u_x - R(\nabla_x^2 \cdot, J_u u_x) u_x \quad (\because \text{(iv)}, \text{(vi)}) \\ &= \nabla_x \{ R(J_u u_x, u_x) \nabla_x \cdot \} - 2R(J_u \nabla_x u_x, u_x) \nabla_x, \quad (\because \text{(3.14)}) \end{aligned} \quad (3.41)$$

$$\begin{aligned} 4J_u (\nabla_x \Phi_1) \nabla_x &= 4J_u [\nabla_x \{ R(\nabla_x \cdot, u_x) u_x \} - R(\nabla_x^2 \cdot, u_x) u_x] \\ &= 4 \{ R(\nabla_x \cdot, \nabla_x u_x) J_u u_x + R(\nabla_x \cdot, u_x) J_u \nabla_x u_x \} \quad (\because \text{(iv)}) \\ &= 4R(J_u \nabla_x u_x, u_x) \nabla_x + 2A_2 \nabla_x + 2A_3 \nabla_x. \quad (\because \text{(3.28)}, \text{(3.29)}) \end{aligned} \quad (3.42)$$

Substituting (3.40)-(3.42) into (3.39), we have

$$\begin{aligned} & -\frac{e_1}{2} [J_u \nabla_x^4, \Phi_1 \nabla_x^{-2}] \\ &= -e_1 R(\nabla_x^2 \cdot, J_u u_x) u_x - \frac{e_1}{2} \nabla_x \{ R(J_u u_x, u_x) \nabla_x \cdot \} - e_1 R(J_u \nabla_x u_x, u_x) \nabla_x \\ &\quad - (e_1 A_2 + e_1 A_3) \nabla_x \end{aligned}$$

$$-3e_1 J_u(\nabla_x^2 \Phi_1) - 2e_1 J_u(\nabla_x^3 \Phi_1) \nabla_x^{-1} - \frac{e_1}{2} J_u(\nabla_x^4 \Phi_1) \nabla_x^{-2}. \quad (3.43)$$

Next, we consider the second term of the RHS of (3.38). To begin with, we see

$$\begin{aligned} [J_u \nabla_x^4, \Phi_2 \nabla_x^{-2}] &= (J_u \Phi_2 - \Phi_2 J_u) \nabla_x^2 + 4J_u(\nabla_x \Phi_2) \nabla_x + 6J_u(\nabla_x^2 \Phi_2) \\ &\quad + 4J_u(\nabla_x^3 \Phi_2) \nabla_x^{-1} + J_u(\nabla_x^4 \Phi_2) \nabla_x^{-2}. \end{aligned} \quad (3.44)$$

In the same way as above, we deduce

$$\begin{aligned} (J_u \Phi_2 - \Phi_2 J_u) \nabla_x^2 &= J_u R(J_u u_x, u_x) J_u \nabla_x^2 - R(J_u u_x, u_x) J_u J_u \nabla_x^2 \\ &= 0, \quad (\because \text{iv}) \end{aligned} \quad (3.45)$$

$$\begin{aligned} 4J_u(\nabla_x \Phi_2) \nabla_x &= 4J_u \{ \nabla_x(\Phi_2 \nabla_x \cdot) - \Phi_2 \nabla_x^2 \} \\ &= 4J_u \{ R(J_u \nabla_x u_x, u_x) J_u \nabla_x + R(J_u u_x, \nabla_x u_x) J_u \nabla_x \} \\ &= -8R(J_u \nabla_x u_x, u_x) \nabla_x. \quad (\because \text{iv}, \text{vi}) \end{aligned} \quad (3.46)$$

Thus, substituting (3.45) and (3.46) into (3.44), we have

$$\begin{aligned} \frac{e_2}{8} [J_u \nabla_x^4, \Phi_2 \nabla_x^{-2}] &= -e_2 R(J_u \nabla_x u_x, u_x) \nabla_x \\ &\quad + \frac{3e_2}{4} J_u(\nabla_x^2 \Phi_2) + \frac{e_2}{2} J_u(\nabla_x^3 \Phi_2) \nabla_x^{-1} + \frac{e_2}{8} J_u(\nabla_x^4 \Phi_2) \nabla_x^{-2}. \end{aligned} \quad (3.47)$$

Therefore, by collecting (3.38), (3.43), and (3.47), we deduce

$$\begin{aligned} [a J_u \nabla_x^4, \Phi \nabla_x^{-2}] \nabla_x^k u_x &= -e_1 R(\nabla_x^2 \nabla_x^k u_x, J_u u_x) u_x - \frac{e_1}{2} \nabla_x \{ R(J_u u_x, u_x) \nabla_x \nabla_x^k u_x \} \\ &\quad + (-e_1 - e_2) R(J_u \nabla_x u_x, u_x) \nabla_x \nabla_x^k u_x - (e_1 A_2 + e_1 A_3) \nabla_x \nabla_x^k u_x \\ &\quad - 3e_1 J_u(\nabla_x^2 \Phi_1) \nabla_x^k u_x - 2e_1 J_u(\nabla_x^3 \Phi_1) \nabla_x^{k-1} u_x - \frac{e_1}{2} J_u(\nabla_x^4 \Phi_1) \nabla_x^{k-2} u_x \\ &\quad + \frac{3e_2}{4} J_u(\nabla_x^2 \Phi_2) \nabla_x^k u_x + \frac{e_2}{2} J_u(\nabla_x^3 \Phi_2) \nabla_x^{k-1} u_x + \frac{e_2}{8} J_u(\nabla_x^4 \Phi_2) \nabla_x^{k-2} u_x \\ &= -e_1 R(\nabla_x^2 \nabla_x^k u_x, J_u u_x) u_x - \frac{e_1}{2} \nabla_x \{ R(J_u u_x, u_x) \nabla_x \nabla_x^k u_x \} \\ &\quad + (-e_1 - e_2) R(J_u \nabla_x u_x, u_x) \nabla_x \nabla_x^k u_x - (e_1 A_2 + e_1 A_3) \nabla_x \nabla_x^k u_x \\ &\quad + \mathcal{O} \left(\sum_{m=0}^k |\nabla_x^m u_x|_h \right). \end{aligned}$$

This together with $\nabla_x^k u_x = \Lambda + \mathcal{O}(|\nabla_x^{k-2} u_x|_h)$ concludes

$$\begin{aligned} [a J_u \nabla_x^4, \Phi \nabla_x^{-2}] \nabla_x^k u_x &= -e_1 R(\nabla_x^2 V_k, J_u u_x) u_x - \frac{e_1}{2} \nabla_x \{ R(J_u u_x, u_x) \nabla_x V_k \} \\ &\quad + (-e_1 - e_2) R(J_u \nabla_x u_x, u_x) \nabla_x V_k - (e_1 A_2 + e_1 A_3) \nabla_x V_k \\ &\quad + \mathcal{O} \left(\sum_{m=0}^k |\nabla_x^m u_x|_h \right). \end{aligned} \quad (3.48)$$

Finally, combining (3.37) and (3.48), we derive

$$\begin{aligned} \nabla_t V_k &= \{ -\varepsilon \nabla_x^4 + a J_u \nabla_x^4 + \lambda J_u \nabla_x^2 \} V_k \\ &\quad + \varepsilon \mathcal{O}(|\nabla_x^{k+2} u_x|_h + |\nabla_x^{k+1} u_x|_h) \end{aligned}$$

$$\begin{aligned}
& + (d_1 + e_1) R(\nabla_x^2 V_k, J_u u_x) u_x + \left(d_2 + \frac{e_1}{2}\right) \nabla_x \{R(J_u u_x, u_x) \nabla_x V_k\} \\
& + (d_3 + e_1 + e_2) R(J_u \nabla_x u_x, u_x) \nabla_x V_k + d_4 R(\nabla_x u_x, u_x) J_u \nabla_x V_k \\
& + \{d_5 A_1 + (d_6 + e_1) A_2 + (d_7 + e_1) A_3\} \nabla_x V_k \\
& + \mathcal{O} \left(\sum_{m=0}^k |\nabla_x^m u_x|_h + |\nabla_x^{k-2} u_x|_h \sum_{m=0}^4 |\nabla_x^m u_x|_h \right). \tag{3.49}
\end{aligned}$$

We go back to the estimate for (3.6). Using (3.49), we have

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \|V_k\|_{L^2}^2 &= \int_{\mathbb{T}} h(\nabla_t V_k, V_k) dx \\
&= -\varepsilon \int_{\mathbb{T}} h(\nabla_x^4 V_k, V_k) dx + \varepsilon \int_{\mathbb{T}} h(\mathcal{O}(|\nabla_x^{k+2} u_x|_h + |\nabla_x^{k+1} u_x|_h), V_k) dx \\
&\quad + \int_{\mathbb{T}} h(\{a J_u \nabla_x^4 + \lambda J_u \nabla_x^2\} V_k, V_k) dx \\
&\quad + (d_1 + e_1) \int_{\mathbb{T}} h(R(\nabla_x^2 V_k, J_u u_x) u_x, V_k) dx \\
&\quad + \left(d_2 + \frac{e_1}{2}\right) \int_{\mathbb{T}} h(\nabla_x \{R(J_u u_x, u_x) \nabla_x V_k\}, V_k) dx \\
&\quad + (d_3 + e_1 + e_2) \int_{\mathbb{T}} h(R(J_u \nabla_x u_x, u_x) \nabla_x V_k, V_k) dx \\
&\quad + d_4 \int_{\mathbb{T}} h(R(\nabla_x u_x, u_x) J_u \nabla_x V_k, V_k) dx \\
&\quad + \int_{\mathbb{T}} h(\{d_5 A_1 + (d_6 + e_1) A_2 + (d_7 + e_1) A_3\} \nabla_x V_k, V_k) dx \\
&\quad + \int_{\mathbb{T}} h \left(\mathcal{O} \left(\sum_{m=0}^k |\nabla_x^m u_x|_h + |\nabla_x^{k-2} u_x|_h \sum_{m=0}^4 |\nabla_x^m u_x|_h \right), V_k \right) dx.
\end{aligned}$$

We compute each term of the above separately. By integrating by parts, we obtain

$$\begin{aligned}
\int_{\mathbb{T}} h(J_u \nabla_x^4 V_k, V_k) dx &= \int_{\mathbb{T}} h(J_u \nabla_x^2 V_k, \nabla_x^2 V_k) dx = 0, \\
\int_{\mathbb{T}} h(J_u \nabla_x^2 V_k, V_k) dx &= - \int_{\mathbb{T}} h(J_u \nabla_x V_k, \nabla_x V_k) dx = 0.
\end{aligned}$$

In the same way, we integrate by part using (i) and (ii) to show

$$\begin{aligned}
& \int_{\mathbb{T}} h(\nabla_x \{R(J_u u_x, u_x) \nabla_x V_k\}, V_k) dx \\
&= - \int_{\mathbb{T}} h(R(J_u u_x, u_x) \nabla_x V_k, \nabla_x V_k) dx \\
&= - \int_{\mathbb{T}} h(R(\nabla_x V_k, \nabla_x V_k) J_u u_x, u_x) dx \quad (\because \text{(ii)}) \\
&= 0. \quad (\because \text{(i)})
\end{aligned}$$

Since $k \geq 4$, we use the Sobolev embedding, the Cauchy-Schwartz inequality, and (3.5) to deduce

$$\begin{aligned} & \int_{\mathbb{T}} h \left(\mathcal{O} \left(\sum_{m=0}^k |\nabla_x^m u_x|_h + |\nabla_x^{k-2} u_x|_h \sum_{m=0}^4 |\nabla_x^m u_x|_h \right), V_k \right) dx \\ & \leq C \|u_x\|_{H^k} \|V_k\|_{L^2} \\ & \leq C (N_k(u))^2. \end{aligned}$$

Using the integration by parts, the Young inequality $AB \leq A^2/2 + B^2/2$ for any $A, B \geq 0$, $\nabla_x^k u_x = V_k + \mathcal{O}(|\nabla_x^{k-2} u_x|_h)$, $\varepsilon \leq 1$, and (3.5), we deduce

$$\begin{aligned} & -\varepsilon \int_{\mathbb{T}} h(\nabla_x^4 V_k, V_k) dx + \varepsilon \int_{\mathbb{T}} h(\mathcal{O}(|\nabla_x^{k+2} u_x|_h + |\nabla_x^{k+1} u_x|_h), V_k) dx \\ & = -\varepsilon \int_{\mathbb{T}} h(\nabla_x^2 V_k, \nabla_x^2 V_k) dx \\ & \quad + \varepsilon \int_{\mathbb{T}} h(\mathcal{O}(|\nabla_x^{k+2} u_x|_h + |\nabla_x^{k+1} u_x|_h), \nabla_x^k u_x + \mathcal{O}(|\nabla_x^{k-2} u_x|_h)) dx \\ & \leq -\varepsilon \|\nabla_x^2 V_k\|_{L^2}^2 + \varepsilon C \|\nabla_x^{k+2} u_x\|_{L^2} (\|\nabla_x^k u_x\|_{L^2} + \|\nabla_x^{k-1} u_x\|_{L^2}) + C \|u_x\|_{H^k}^2 \\ & \leq -\varepsilon \|\nabla_x^2 V_k\|_{L^2}^2 + \frac{\varepsilon}{2} \|\nabla_x^{k+2} u_x\|_{L^2}^2 + \frac{\varepsilon C^2}{2} (\|\nabla_x^k u_x\|_{L^2} + \|\nabla_x^{k-1} u_x\|_{L^2})^2 + C \|u_x\|_{H^k}^2 \\ & \leq -\varepsilon \|\nabla_x^2 V_k\|_{L^2}^2 + \frac{\varepsilon}{2} \|\nabla_x^2 V_k\|_{L^2}^2 + C \|u_x\|_{H^k}^2 \\ & \leq -\frac{\varepsilon}{2} \|\nabla_x^2 V_k\|_{L^2}^2 + C (N_k(u))^2. \end{aligned}$$

Note that $R(\nabla_x u_x, u_x) J_u$ is symmetric. Indeed,

$$\begin{aligned} h(R(\nabla_x u_x, u_x) J_u Y, Z) &= h(R(J_u Y, Z) \nabla_x u_x, u_x) \quad (\because \text{(ii)}) \\ &= h(R(J_u Z, Y) \nabla_x u_x, u_x) \quad (\because \text{(vi)}) \\ &= h(R(\nabla_x u_x, u_x) J_u Z, Y) \quad (\because \text{(ii)}) \\ &= h(Y, R(\nabla_x u_x, u_x) J_u Z) \end{aligned}$$

for any $Y, Z \in \Gamma(u^{-1}TN)$. Therefore, the integration by parts implies

$$\begin{aligned} & \int_{\mathbb{T}} h(R(\nabla_x u_x, u_x) J_u \nabla_x^{k+1} u_x, \nabla_x^k u_x) dx \\ & = -\frac{1}{2} \int_{\mathbb{T}} h(R(\nabla_x^2 u_x, u_x) J_u \nabla_x^k u_x, \nabla_x^k u_x) dx \\ & \quad - \frac{1}{2} \int_{\mathbb{T}} h(R(\nabla_x u_x, \nabla_x u_x) J_u \nabla_x^k u_x, \nabla_x^k u_x) dx \\ & \leq C \|u_x\|_{H^k}^2 \\ & \leq C (N_k(u))^2. \end{aligned}$$

As we observed (3.25), A_i , $i = 1, 2, 3$, are symmetric. Thus, in the same way as above, the integration by parts shows

$$\int_{\mathbb{T}} h(A_i \nabla_x V_k, V_k) dx = -\frac{1}{2} \int_{\mathbb{T}} h((\nabla_x A_i) V_k, V_k) dx \leq C (N_k(u))^2$$

for each $i = 1, 2, 3$. Hence, we have

$$\int_{\mathbb{T}} h(\{d_5 A_1 + (d_6 + e_1) A_2 + (d_7 + e_1) A_3\} \nabla_x V_k, V_k) dx \leq C(N_k(u))^2.$$

Collecting them, we derive

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|V_k\|_{L^2}^2 &\leq -\frac{\varepsilon}{2} \|\nabla_x^2 V_k\|_{L^2}^2 + (d_1 + e_1) \int_{\mathbb{T}} h(R(\nabla_x^2 V_k, J_u u_x) u_x, V_k) dx \\ &\quad + (d_3 + e_1 + e_2) \int_{\mathbb{T}} h(R(J_u \nabla_x u_x, u_x) \nabla_x V_k, V_k) dx + C(N_k(u))^2. \end{aligned} \quad (3.50)$$

To cancel the second and the third term of the RHS of above, we set e_1 and e_2 so that

$$\begin{aligned} e_1 &= -d_1 = a - b, \\ e_2 &= -d_3 - e_1 = \left(-k - \frac{1}{2}\right) a + \left(-k + \frac{5}{2}\right) b + (-2k - 1) c. \end{aligned}$$

Therefore, we derive

$$\frac{1}{2} \frac{d}{dt} \|V_k^\varepsilon\|_{L^2}^2 \leq -\frac{\varepsilon}{2} \|\nabla_x^2 V_k^\varepsilon\|_{L^2}^2 + C(N_k(u^\varepsilon))^2. \quad (3.51)$$

Concerning the uniform estimate for $\{N_k(u^\varepsilon)\}_{\varepsilon \in (0,1]}$, it remains to consider the energy estimate for $\|u_x^\varepsilon\|_{H^{k-1}}^2$. However, by using the integration by parts, the Sobolev embedding, and the Cauchy-Schwartz inequality repeatedly, it is now easy to show

$$\frac{1}{2} \frac{d}{dt} \|u_x^\varepsilon\|_{H^{k-1}}^2 \leq -\frac{\varepsilon}{2} \sum_{m=0}^{k-1} \|\nabla_x^{m+2} u_x^\varepsilon\|_{L^2}^2 + C(N_k(u^\varepsilon))^2. \quad (3.52)$$

Therefore, from (3.51) and (3.52), we conclude that there exists a positive constant C depending on $a, b, c, k, \lambda, \|u_{0x}\|_{H^4}$ and not on ε such that

$$\frac{d}{dt} (N_k(u^\varepsilon))^2 = \frac{d}{dt} (\|u_x^\varepsilon\|_{H^{k-1}}^2 + \|V_k^\varepsilon\|_{L^2}^2) \leq C(N_k(u^\varepsilon))^2$$

on the time-interval $[0, T_\varepsilon^*]$. This implies $(N_k(u^\varepsilon(t)))^2 \leq (N_k(u_0))^2 e^{Ct}$ for any $t \in [0, T_\varepsilon^*]$. Thus, by the definition of T_ε^* , there holds

$$4(N_4(u_0))^2 = (N_4(u^\varepsilon(T_\varepsilon^*)))^2 \leq (N_4(u_0))^2 e^{C_4 T_\varepsilon^*}$$

with $C_4 > 0$ which depends on $a, b, c, \lambda, \|u_{0x}\|_{H^4}$ and not on ε . This shows $e^{C_4 T_\varepsilon^*} \geq 4$ and hence $T_\varepsilon^* \geq (\log 4)/C_4$ holds. Therefore, if we set $T = (\log 4)/C_4$, it follows that $T_\varepsilon^* \geq T$ for any $\varepsilon \in (0, 1]$ and $\{N_k(u^\varepsilon)\}_{\varepsilon \in (0,1]}$ is bounded in $L^\infty(0, T)$.

As stated before, this shows that $\{u_x\}_{\varepsilon \in (0,1]}$ is bounded in $L^\infty(0, T; H^k(\mathbb{T}; TN))$. Hence the standard compactness argument shows the existence of a map $u \in C([0, T] \times \mathbb{T}; N)$ and a subsequence $\{u^{\varepsilon(j)}\}_{j=1}^\infty$ of $\{u^\varepsilon\}_{\varepsilon \in (0,1]}$ that satisfy

$$\begin{aligned} u_x^{\varepsilon(j)} &\rightarrow u_x \quad \text{in } C([0, T]; H^{k-1}(\mathbb{T}; TN)), \\ u_x^{\varepsilon(j)} &\rightarrow u_x \quad \text{in } L^\infty(0, T; H^k(\mathbb{T}; TN)) \quad \text{weakly star} \end{aligned}$$

as $j \rightarrow \infty$, and this u is smooth and solves (1.1)-(1.2).

Finally, in the general case where $u_0 \in C(\mathbb{T}; N)$ and $u_{0x} \in H^k(\mathbb{T}; TN)$, it suffices to modify the above argument slightly by taking a sequence $\{u_0^i\}_{i=1}^\infty \subset C^\infty(\mathbb{T}; N)$ such that

$$u_{0x}^i \rightarrow u_{0x} \quad \text{in } H^k(\mathbb{T}; TN) \quad (3.53)$$

as $i \rightarrow \infty$. We omit the detail, because the argument of this part is the same as that in [22]. \square

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